A strong nominal semantics for fixed-point constraints

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What are fixed-point constraints?

Nominal Syntax

The set of nominal terms $T(\Sigma, \mathbb{A}, \mathbb{V})$ are defined inductively by the following grammar:

$$t ::= a \mid \pi \cdot X \mid \mathtt{f}(t_1, \ldots, t_n) \mid [a]t$$

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- a range over an infinite countable set of atoms $\mathbb{A}$ (object-level);
- X range over an infinite countable set of variables $\mathbb{V}$ (meta-level);
- $\pi \cdot X$ are called suspensions. $\mathtt{Id} \cdot X$ is represented by X;
- π range over finite permutations on $\mathbb{A}$, i.e. bijections $\mathbb{A} \to \mathbb{A}$ with $\mathtt{dom}(\pi) := \{a \in \mathbb{A} \mid \pi(a) \neq a\}$ is finite;

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- π range over finite permutations on $\mathbb{A}$, i.e. bijections $\mathbb{A} \rightarrow \mathbb{A}$ with $\mathtt{dom}(\pi) := \{a \in \mathbb{A} \mid \pi(a) \neq a\}$ is finite;
- $\mathtt{f}$ range over a finite signature Σ;
- $[a]t$ denotes the abstraction of the atom a over the term t; it represents "$x.e$" or "$x.\phi$" in expressions like "$\lambda x.e$" or "$\forall x.\phi$".

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- Originally, nominal theory was defined using freshness constraints $a \# t$, which generalizes that a is not a free name in t [UPG04; Gab09; GM09].
- Fixed-points were inspired by the following equivalence [GP02; And13]

$$a \# x \iff \text{И} c.(a\ c) \cdot x = x,$$

where the quantifier new (И) quantifies over fresh names, that is, a is fresh for x iff for any fresh atom c, the swapping $(a\ c)$ fixes x.

Why work with fixed-point constraints?

- Nominal unification involves finding substitutions that make two nominal terms equal, originally defined using freshness constraints [UPG04].
- Nominal unification is unitary with freshness constraints but loses this property with equational theories like commutativity C [Aya+17; Aya+19].

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- Nominal unification involves finding substitutions that make two nominal terms equal, originally defined using freshness constraints [UPG04].
- Nominal unification is unitary with freshness constraints but loses this property with equational theories like commutativity C [Aya+17; Aya+19].
- For instance, $(a\ b) \cdot X =^?_{\mathtt{C}} X$ has as solution $\langle \{a\#X, b\#X\}, \mathtt{id} \rangle$. However, there are infinite solutions to the problem:

$$[X \mapsto a + b], [X \mapsto (a + b) + (a + b)], [X \mapsto \mathtt{f}(a + b)], \ldots$$

- This property is recovered with the introduction of fixed-points [AFN20]
- $\langle \{(a\ b) \curlywedge X\}, \mathtt{id} \rangle$ solves it and recover all the lost solutions.

Derivation rules for fixed-points

$$\frac{\pi(a) = a}{\Upsilon \vdash \pi \curlywedge a} \,(\curlywedge \mathtt{a}) \qquad \frac{\Upsilon \vdash \pi \curlywedge t_1 \ \cdots \ \Upsilon \vdash \pi \curlywedge t_n}{\Upsilon \vdash \pi \curlywedge \mathtt{f}(t_1, \ldots, t_n)} \,(\curlywedge \mathtt{f})$$

$$\frac{\mathtt{dom}(\pi^{\pi'^{-1}}) \subseteq \mathtt{dom}(\mathtt{perm}(\Upsilon|_X))}{\Upsilon \vdash \pi \curlywedge \pi' \cdot X} \,(\curlywedge \mathbf{var}) \qquad \frac{\Upsilon, \overline{(c_1\ c_2) \curlywedge \mathtt{Var}(t)} \vdash \pi \curlywedge (a\ c_1) \cdot t}{\Upsilon \vdash \pi \curlywedge [a]t} \,(\curlywedge \mathbf{abs})$$

Figure 1: Derivation rules for fixed-points; c_1, c_2 are fresh names.

- Fixed-point contexts (Υ) contain primitive constraints of the form $\pi \curlywedge X$.

A simple but important example

With the context $\Upsilon = \{(a_1\ a_2) \curlywedge X_1, (a_3\ a_4) \curlywedge X_1\}$ we can derive

$$\dfrac{\dfrac{\{a_1, a_3\} \subseteq \{a_1, a_2, a_3, a_4\}}{\mathtt{dom}((a_1\ a_3)) \subseteq \mathtt{dom}(\mathtt{perm}(\Upsilon|_{X_1}))}}{\{(a_1\ a_2) \curlywedge X_1, (a_3\ a_4) \curlywedge X_1\} \vdash (a_1\ a_3) \curlywedge X_1}\ (\curlywedge\mathsf{var})$$

Nominal theory

- An equality constraint is a pair $t = u$ where t and u are nominal terms.
- An axiom is an equality judgement $\Upsilon \vdash t = u$.
- A (nominal) theory $\mathsf{T} = (\Sigma, Ax)$ consists of a signature Σ and a (possibly) infinite set of axioms Ax.

For example,

- $\mathtt{CORE}_{\curlywedge}$ represents a theory with no axioms $\mathsf{T} = (\Sigma, \emptyset)$.
- $\mathsf{C} = (\Sigma, \{\ \vdash X + Y = Y + X\})$ represents a commutative theory.

Derivation rules for equality via fixed-point constraints

$$\frac{}{\Upsilon \vdash t = t}\ (\mathsf{refl}) \qquad \frac{\Upsilon \vdash t = u}{\Upsilon \vdash u = t}\ (\mathsf{symm}) \qquad \frac{\Upsilon \vdash t = u \qquad \Upsilon \vdash u = v}{\Upsilon \vdash t = v}\ (\mathsf{tran})$$

$$\frac{\Upsilon \vdash (\pi \cdot \Upsilon')\sigma}{\Upsilon \vdash \pi \cdot t\sigma = \pi \cdot u\sigma}\ (\mathsf{ax}_{\Upsilon' \vdash t = u}) \qquad \frac{\Upsilon \vdash t = u}{\Upsilon \vdash [a]t = [a]u}\ (\mathsf{cong}[])$$

$$\frac{\Upsilon \vdash t = u}{\Upsilon \vdash \mathtt{f}(\ldots, t, \ldots) = \mathtt{f}(\ldots, u, \ldots)}\ (\mathsf{congf})$$

$$\frac{\Upsilon, \pi \curlywedge X \vdash t = u \quad (\mathtt{dom}(\pi) \subseteq \mathtt{dom}(\mathtt{perm}(\Upsilon|_X)))}{\Upsilon \vdash t = u}\ (\mathsf{fr})$$

$$\frac{\Upsilon, \overline{(c_1\ c_2) \curlywedge \mathtt{Var}(t)} \vdash (a\ c_1) \curlywedge t \qquad \Upsilon, \overline{(d_1\ d_2) \curlywedge \mathtt{Var}(t)} \vdash (b\ d_1) \curlywedge t}{\Upsilon \vdash (a\ b) \cdot t = t}\ (\mathsf{perm})$$

Figure 2: Derivation rules for equality; c_1, c_2, d_1, d_2 are fresh names.

- $(\pi \cdot \Upsilon')\sigma = \{\pi'^{\pi} \curlywedge \pi \cdot X\sigma \mid \pi' \curlywedge X \in \Upsilon'\}$, where σ is a substitution.

Semantics

Nominal sets

Suppose $\mathscr{X} = (|\mathscr{X}|, \cdot)$ is a $\texttt{Perm}(\mathbb{A})$-set, i.e. a set equipped with a permutation action.

Definition

- The support of an element $x \in |\mathscr{X}|$, denoted by $\mathtt{supp}(x)$, is the least finite atom set that supports x, that is, for all permutations π,

$$(\forall a \in \mathtt{supp}(x).\, \pi(a) = a) \implies \pi \cdot x = x. \tag{1}$$

- Additionally, $\mathtt{supp}(x)$ is strong if it also satisfies the converse of (1).
- $\mathscr{X}$ is a nominal set iff all elements have a finite support. Similarly, $\mathscr{X}$ is a strong nominal set iff all elements have a strong finite support.

Examples

Example

1. The set $\mathbb{A}$ with the action $\pi \cdot a = \pi(a)$ for every $a \in \mathbb{A}$ and $\pi \in \mathtt{Perm}(\mathbb{A})$; $\mathtt{supp}(a) = \{a\}$ for all $a \in \mathbb{A}$.

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2. The set $\mathcal{P}_{\texttt{fin}}(\mathbb{A}) = \{B \subset \mathbb{A} \mid B \text{ is finite}\}$ is a nominal set when equipped with the action $\pi \cdot B = \{\pi \cdot a \mid a \in B\}$ for every $B \in \mathcal{P}_{\texttt{fin}}(\mathbb{A})$ and $\pi \in \texttt{Perm}(\mathbb{A})$; $\texttt{supp}(B) = B$ for all $B \in \mathcal{P}_{\texttt{fin}}(\mathbb{A})$.
   - $\mathcal{P}_{\texttt{fin}}(\mathbb{A})$ is NOT strong because for $B = \{a, b\}$ and $\pi = (a\ b)$ we have $\pi \cdot B = B$ but $\pi(a) \neq a$.

Examples

Example

3. The set of all ground nominal terms $T(\Sigma, \mathbb{A}, \emptyset)$ with the usual permutation action forms a strong nominal set; $\mathtt{supp}(g) = \mathtt{atm}(g)$ for all $g \in T(\Sigma, \mathbb{A}, \emptyset)$, where $\mathtt{atm}(g)$ is the set of all atoms that occur in g.
4. Quotienting $T(\Sigma, \mathbb{A}, \emptyset)$ by the relation $g \sim g'$ iff $\vdash_{\mathsf{T}} g = g'$, then the set $T(\Sigma, \mathbb{A}, \emptyset)/_{\sim}$ is a nominal set. We usually denote it just by $\mathbb{F}(\mathsf{T}, \Sigma)$; $\mathtt{supp}(\overline{g}) = \bigcap\{\mathtt{supp}(g') \mid g' \in \overline{g}\}$ for all $\overline{g} \in \mathbb{F}(\mathsf{T}, \Sigma)$.

For C, the nominal set $\mathbb{F}(\mathsf{C}, \Sigma)$ is not strong.

Σ-algebras

Definition

Given a signature Σ, a (strong) Σ-algebra $\mathfrak{A}$ consists of:

1. A domain (strong) nominal set $\mathscr{A} = (|\mathscr{A}|, \cdot)$.
2. An equivariant map $\mathtt{atom}^{\mathfrak{A}} : \mathbb{A} \to |\mathscr{A}|$ to interpret atoms;
3. An equivariant map $\mathtt{abs}^{\mathfrak{A}} : \mathbb{A} \times |\mathscr{A}| \to |\mathscr{A}|$ to interpret abstractions, such that $a \notin \mathtt{supp}(\mathtt{abs}^{\mathfrak{A}}(a, x))$ for any $a \in \mathbb{A}$ and $x \in |\mathscr{A}|$.
4. An equivariant map $f^{\mathfrak{A}} : |\mathscr{A}|^n \to |\mathscr{A}|$ for each term-former $\mathtt{f} : n$ in Σ.

Interpreting nominal terms

Definition

- A valuation ς in $\mathfrak{A}$ maps variables $X \in \mathbb{V}$ to elements $\varsigma(X) \in |\mathscr{A}|$.
- The interpretation of a nominal term t, denoted by $[\![t]\!]^{\mathfrak{A}}_{\varsigma}$ is defined inductively by:

$$[\![a]\!]^{\mathfrak{A}}_{\varsigma} = \mathtt{atom}^{\mathfrak{A}}(a) \qquad [\![\pi \cdot X]\!]^{\mathfrak{A}}_{\varsigma} = \pi \cdot \varsigma(X)$$

$$[\![\mathsf{f}(t_1, \ldots, t_n)]\!]^{\mathfrak{A}}_{\varsigma} = f^{\mathfrak{A}}([\![t_1]\!]^{\mathfrak{A}}_{\varsigma}, \ldots, [\![t_n]\!]^{\mathfrak{A}}_{\varsigma}) \qquad [\![[a]t]\!]^{\mathfrak{A}}_{\varsigma} = \mathtt{abs}^{\mathfrak{A}}(a, [\![t]\!]^{\mathfrak{A}}_{\varsigma}).$$

Validity and models

Definition

Let $\mathfrak{A}$ be a (strong) Σ-algebra and ς a valuation on $\mathfrak{A}$.

- $[\![\Upsilon]\!]^{\mathfrak{A}}_{\varsigma}$ is valid iff $\pi \cdot [\![X]\!]^{\mathfrak{A}}_{\varsigma} = [\![X]\!]^{\mathfrak{A}}_{\varsigma}$ for each $\pi \curlywedge X \in \Upsilon$.
- $[\![\Upsilon \vdash \pi \curlywedge t]\!]^{\mathfrak{A}}_{\varsigma}$ is valid iff $[\![\Upsilon]\!]^{\mathfrak{A}}_{\varsigma}$ (valid) implies $\pi \cdot [\![t]\!]^{\mathfrak{A}}_{\varsigma} = [\![t]\!]^{\mathfrak{A}}_{\varsigma}$.
- $[\![\Upsilon \vdash s = t]\!]^{\mathfrak{A}}_{\varsigma}$ is valid iff $[\![\Upsilon]\!]^{\mathfrak{A}}_{\varsigma}$ (valid) implies $[\![s]\!]^{\mathfrak{A}}_{\varsigma} = [\![t]\!]^{\mathfrak{A}}_{\varsigma}$.

Definition

Let $\mathsf{T} = (\Sigma, Ax)$ be a theory. A (strong) model of T is a (strong) Σ-algebra $\mathfrak{A}$ such that for every valuation ς in $\mathfrak{A}$ we have that

$$[\![\Upsilon \vdash t = u]\!]^{\mathfrak{A}}_{\varsigma} \text{ is valid for every axiom } \Upsilon \vdash t = u \in Ax.$$

Soundness

Question: Is soundness true?

Suppose $\mathsf{T} = (\Sigma, Ax)$ is a theory, $\mathfrak{A}$ is a Σ-algebra which is a model of T, and ς is a valuation on $\mathfrak{A}$. Then:

1. If $\Upsilon \vdash \pi \curlywedge t$ then $[\![\Upsilon \vdash \pi \curlywedge t]\!]^{\mathfrak{A}}_{\varsigma}$ is valid.
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No! The nominal set semantics for nominal theories via fixed-point constraints fails to be sound. The culprit is the rule

$$\frac{\mathtt{dom}(\pi^{\pi'^{-1}}) \subseteq \mathtt{dom}(\mathtt{perm}(\Upsilon|_X))}{\Upsilon \vdash \pi \curlywedge \pi' \cdot X} \;(\curlywedge\mathbf{var})$$

We're going to present a Σ-algebra $\mathfrak{A}$, a valuation ς and a derivation $\Upsilon \vdash \pi \curlywedge \pi' \cdot X$ using $(\curlywedge\mathbf{var})$ such that $[\![\Upsilon \vdash \pi \curlywedge \pi' \cdot X]\!]^{\mathfrak{A}}_{\varsigma}$ is not valid.

A counter-example to soundness

Consider the domain of $\mathfrak{A}$ as the nominal set $\mathcal{P}_{\mathtt{fin}}(\mathbb{A})$.

- Fix enumerations of $\mathbb{V} = \{X_1, X_2, \ldots\}$ and $\mathbb{A} = \{a_1, a_2, \ldots\}$.
- Define the valuation $\varsigma(X_i) := \{a_i, a_{i+1}\}$. Then $\varsigma(X_1) = \{a_1, a_2\}, \varsigma(X_2) = \{a_2, a_3\}$ and so on.

Consider the derivation $\{(a_1\ a_2) \curlywedge X_1, (a_3\ a_4) \curlywedge X_1\} \vdash (a_1\ a_3) \curlywedge X_1$ from before. Then

- $[\![\Upsilon]\!]^{\mathfrak{A}}_{\varsigma}$ is valid because $(a_1\ a_2) \cdot [\![X_1]\!]^{\mathfrak{A}}_{\varsigma} = [\![X_1]\!]^{\mathfrak{A}}_{\varsigma}$ and $(a_3\ a_4) \cdot [\![X_1]\!]^{\mathfrak{A}}_{\varsigma} = [\![X_1]\!]^{\mathfrak{A}}_{\varsigma}$.
- However, $(a_1\ a_3) \cdot [\![X_1]\!]^{\mathfrak{A}}_{\varsigma} \neq [\![X_1]\!]^{\mathfrak{A}}_{\varsigma}$ because

$$(a_1\ a_3) \cdot [\![X_1]\!]^{\mathfrak{A}}_{\varsigma} = \{a_3, a_2\} \neq \{a_1, a_2\} = [\![X_1]\!]^{\mathfrak{A}}_{\varsigma}.$$

Soundness for strong models

By restricting the semantics to the class of strong nominal sets, we obtain a weak version of soundness:

Theorem (Soundness for strong models)

Suppose $\mathsf{T} = (\Sigma, Ax)$ is a theory, $\mathfrak{A}$ is a strong Σ-algebra which is a strong model of T, and ς is a valuation on $\mathfrak{A}$. Then:

1. If $\Upsilon \vdash \pi \curlywedge t$ then $[\![\Upsilon \vdash \pi \curlywedge t]\!]^{\mathfrak{A}}_{\varsigma}$ is valid.
2. If $\Upsilon \vdash_{\mathsf{T}} t = u$ then $[\![\Upsilon \vdash t = u]\!]^{\mathfrak{A}}_{\varsigma}$ is valid.

What about completeness?

Strong theories

- Completeness relies on the following result: for a theory T, the set $\mathbb{F}(\mathsf{T}, \Sigma)$ is strong nominal.
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Definition

- Given a term t, we write $X <_t Y$ if X occurs in t at position p and Y occurs in t at position q with $p <_{lex} q$.
- An axiom $\vdash t = u$ is strong if the following hold:
  1. t and u are first-order terms (i.e., they are built using just function symbols and variables);
  2. $<_t$ and $<_u$ are strict partial orders (we say that t, u are well-ordered);
  3. the order of the variables that occur in t and in u is compatible: if $X <_t Y$ then it is not the case that $Y <_u X$.

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Theorem

If T is a strong theory, then $\mathbb{F}(\mathsf{T}, \Sigma)$ is a strong nominal set.

Examples: Strong and non-strong theories

Example (non-strong)

Condition 1 excludes axioms like $\mathtt{ATOM} = \{\ \vdash a = b\}$. Condition 2 excludes axioms like distributivity $\mathtt{D} = \{\ \vdash X * (Y + Z) = X * Y + X * Z\}$. Condition 3 excludes permutative theories like $\mathtt{C} = \{\ \vdash \mathtt{f}(X, Y) = \mathtt{f}(Y, X)\}$.

Example (strong)

The following axioms (and their combinations) are strong:

- Associativity $\mathtt{A} = \{\ \vdash \mathtt{f}(\mathtt{f}(X, Y), Z) = \mathtt{f}(X, \mathtt{f}(Y, Z))\}$.
- Homomorphism $\mathtt{Hom} = \{\ \vdash \mathtt{h}(X * Y) = \mathtt{h}(X) * \mathtt{h}(Y)\}$.
- Idempotency $\mathtt{I} = \{\ \vdash \mathtt{g}(X, X) = X\}$.
- Neutral element $\mathtt{N} = \{\ \vdash X * 0 = 0\}$.
- Left-/right-projection $\mathtt{Lproj} = \{\ \vdash \mathtt{pl}(X, Y) = X\}$ and $\mathtt{Rproj} = \{\ \vdash \mathtt{pr}(X, Y) = Y\}$.

Recovering Soundness

Strong judgments

- Recall Pitts' equivalence: $a\#x \iff \text{И}c.(a\ c) \cdot x = x$,
- Primitive constraints should've be of the form $\text{И}c.(a\ c) \curlywedge X$ instead of just $\pi \curlywedge X$.

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Definition

- A strong fixed-point context, denoted $\Upsilon_{A,C}$, consists of a finite set with primitive constraints of the form $\text{И}c.(a\ c) \curlywedge X$, where $a \in A$, $c \in C$, and A and C are disjoint sets of atoms.
- A strong fixed-point judgment is of the form $\text{И}\overline{c}.(\Upsilon_{A,\overline{c_0}} \vdash \pi \curlywedge t)$ where $\Upsilon_{A,\overline{c_0}}$ is a strong fixed-point context and $\overline{c_0} \subseteq \overline{c}$. Similarly, a strong α-equality judgment takes the form $\text{И}\overline{c}.(\Upsilon_{A,\overline{c_0}} \vdash s \overset{\curlywedge}{\approx}_\alpha t)$.

Remember the fixed-point derivation rules?

$$\frac{\pi(a) = a}{\Upsilon \vdash \pi \curlywedge a}\ (\curlywedge \mathtt{a}) \qquad \frac{\Upsilon \vdash \pi \curlywedge t_1 \ \cdots \ \Upsilon \vdash \pi \curlywedge t_n}{\Upsilon \vdash \pi \curlywedge \mathtt{f}(t_1, \ldots, t_n)}\ (\curlywedge \mathtt{f})$$

$$\frac{\mathtt{dom}(\pi^{\pi'^{-1}}) \subseteq \mathtt{dom}(\mathtt{perm}(\Upsilon|_X))}{\Upsilon \vdash \pi \curlywedge \pi' \cdot X}\ (\curlywedge \mathtt{var}) \qquad \frac{\Upsilon, \overline{(c_1\ c_2) \curlywedge \mathtt{Var}(t)} \vdash \pi \curlywedge (a\ c_1) \cdot t}{\Upsilon \vdash \pi \curlywedge [a]t}\ (\curlywedge \mathtt{abs})$$

Figure 3: Derivation rules for fixed-points; c_1, c_2 are fresh names.

These are them now. Feel old yet?

$$\frac{\pi(a) = a}{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash \pi \curlywedge a}\ (\curlywedge \mathtt{a}) \qquad \frac{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash \pi \curlywedge t_1 \ \cdots \ \text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash \pi \curlywedge t_n}{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash \pi \curlywedge \mathtt{f}(t_1, \ldots, t_n)}\ (\curlywedge \mathtt{f})$$

$$\frac{\mathtt{dom}(\pi^{\pi'^{-1}}) \setminus \overline{c} \subseteq \mathtt{dom}(\mathtt{perm}(\Upsilon_{A,\overline{c_0}}|_X))}{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash \pi \curlywedge \pi' \cdot X}\ (\curlywedge \mathtt{var}) \qquad \frac{\text{И}\overline{c}, c_1.\Upsilon_{A,\overline{c_0}} \vdash \pi \curlywedge (a\ c_1) \cdot t}{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash \pi \curlywedge [a]t}\ (\curlywedge \mathtt{abs})$$

Figure 4: Derivation rules for strong judgements. Here, $\overline{c}$ denotes a list of distinct atoms $c_1, \ldots, c_n$. In all the rules $\overline{c_0} \subseteq \overline{c}$.

New derivation rules for α-equality

$$\frac{}{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash a \stackrel{\curlywedge}{\approx}_\alpha a}\ (\stackrel{\curlywedge}{\approx}_\alpha \mathtt{a}) \qquad \frac{\mathtt{dom}(\pi'^{-1} \circ \pi) \setminus \overline{c} \subseteq \mathtt{dom}(\mathtt{perm}(\Upsilon_{A,\overline{c_0}}|_X))}{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash \pi \cdot X \stackrel{\curlywedge}{\approx}_\alpha \pi' \cdot X}\ (\stackrel{\curlywedge}{\approx}_\alpha \mathtt{var})$$

$$\frac{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash t_1 \stackrel{\curlywedge}{\approx}_\alpha t'_1 \ \cdots \ \text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash t_n \stackrel{\curlywedge}{\approx}_\alpha t'_n}{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash \mathtt{f}(t_1, \ldots, t_n) \stackrel{\curlywedge}{\approx}_\alpha \mathtt{f}(t'_1, \ldots, t'_n)}\ (\stackrel{\curlywedge}{\approx}_\alpha \mathtt{f})$$

$$\frac{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash t \stackrel{\curlywedge}{\approx}_\alpha t'}{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash [a]t \stackrel{\curlywedge}{\approx}_\alpha [a]t'}\ (\stackrel{\curlywedge}{\approx}_\alpha [\mathtt{a}])$$

$$\frac{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash s \stackrel{\curlywedge}{\approx}_\alpha (a\ b) \cdot t \qquad \text{И}\overline{c}, c_1.\Upsilon_{A,\overline{c_0}} \vdash (a\ c_1) \curlywedge t}{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash [a]s \stackrel{\curlywedge}{\approx}_\alpha [b]t}\ (\stackrel{\curlywedge}{\approx}_\alpha \mathtt{ab})$$

Figure 5: Derivation rules for strong judgements. Here, $\overline{c}$ denotes a list of distinct atoms $c_1, \ldots, c_n$. In all the rules $\overline{c_0} \subseteq \overline{c}$.

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$$\frac{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash t \stackrel{\curlywedge}{\approx}_\alpha t'}{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash [a]t \stackrel{\curlywedge}{\approx}_\alpha [a]t'}\ (\stackrel{\curlywedge}{\approx}_\alpha [\mathtt{a}])$$

$$\frac{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash s \stackrel{\curlywedge}{\approx}_\alpha (a\ b) \cdot t \qquad \text{И}\overline{c}, c_1.\Upsilon_{A,\overline{c_0}} \vdash (a\ c_1) \curlywedge t}{\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash [a]s \stackrel{\curlywedge}{\approx}_\alpha [b]t}\ (\stackrel{\curlywedge}{\approx}_\alpha \mathtt{ab})$$

Figure 5: Derivation rules for strong judgements. Here, $\overline{c}$ denotes a list of distinct atoms $c_1, \ldots, c_n$. In all the rules $\overline{c_0} \subseteq \overline{c}$.

Theorem (Correctness)

$\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash \pi \curlywedge t$ if and only if $\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash \pi \cdot t \stackrel{\curlywedge}{\approx}_\alpha t$, where $\overline{c_0} \subseteq \overline{c}$.

Translation between freshness and fixed-point constraints

Translations:

$$
\begin{array}{rrcl}
[\cdot]_{\curlywedge} : & a\#X & \mapsto & \text{И} c_a.(a\ c_a) \curlywedge X \\
[\cdot]_{\#} : & \text{И} c.(a\ c) \curlywedge X & \mapsto & a\#X.
\end{array}
$$

Theorem A

The following hold, for some $\overline{c}$ (possibly empty):

1. $\Delta \vdash a\#t \iff \text{И}\overline{c}, c_1.[\Delta]_{\curlywedge} \vdash (a\ c_1) \curlywedge t.$
2. $\Delta \vdash s \approx_\alpha t \iff \text{И}\overline{c}.[\Delta]_{\curlywedge} \vdash s \overset{\curlywedge}{\approx}_\alpha t.$

Translation between freshness and fixed-point constraints

Theorem B

The following hold, for $\overline{c_0} \subseteq \overline{c}$:

1. $\text{И}\overline{c}, c_1.\Upsilon_{A,\overline{c_0}} \vdash (a\ c_1) \curlywedge t \iff [\Upsilon_{A,\overline{c_0}}]_{\#} \vdash a \# t.$
2. $\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash s \stackrel{\curlywedge}{\approx}_{\alpha} t \iff [\Upsilon_{A,\overline{c_0}}]_{\#}, \overline{\overline{c} \# \mathtt{Var}(s,t)} \vdash s \approx_{\alpha} t.$

Translation between freshness and fixed-point constraints

Theorem B

The following hold, for $\overline{c_0} \subseteq \overline{c}$:

1. $\text{И}\overline{c}, c_1.\Upsilon_{A,\overline{c_0}} \vdash (a\ c_1) \curlywedge t \iff [\Upsilon_{A,\overline{c_0}}]_\# \vdash a\#t.$
2. $\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash s \overset{\curlywedge}{\approx}_\alpha t \iff [\Upsilon_{A,\overline{c_0}}]_\#, \overline{\overline{c}\#\mathtt{Var}(s,t)} \vdash s \approx_\alpha t.$

- A fragment of the calculus with strong fixed-point constraints is equivalent to the calculus with freshness constraints.
- The judgement $\text{И}c_1, c_2.(a\ c_1) \curlywedge X, (b\ c_2) \curlywedge X \vdash (a\ b) \curlywedge X$ is derivable, and it cannot be translate to a freshness judgement. It remains valid since it is equivalent to $a\#X, b\#X \vdash (a\ b) \cdot X \approx X$.

Translation between freshness and fixed-point constraints

Theorem B

The following hold, for $\overline{c_0} \subseteq \overline{c}$:

1. $\text{И}\overline{c}, c_1.\Upsilon_{A,\overline{c_0}} \vdash (a\ c_1) \curlywedge t \iff [\Upsilon_{A,\overline{c_0}}]_{\#} \vdash a\#t.$
2. $\text{И}\overline{c}.\Upsilon_{A,\overline{c_0}} \vdash s \stackrel{\curlywedge}{\approx}_\alpha t \iff [\Upsilon_{A,\overline{c_0}}]_{\#}, \overline{\overline{c}\#\mathtt{Var}(s,t)} \vdash s \approx_\alpha t.$

- A fragment of the calculus with strong fixed-point constraints is equivalent to the calculus with freshness constraints.
- The judgement $\text{И}c_1, c_2.(a\ c_1) \curlywedge X, (b\ c_2) \curlywedge X \vdash (a\ b) \curlywedge X$ is derivable, and it cannot be translate to a freshness judgement. It remains valid since it is equivalent to $a\#X, b\#X \vdash (a\ b) \cdot X \approx X$.

Theorem C

Nominal sets are sound for strong fixed-point constraints and judgements.

Another way of recovering soundness in nominal sets?

Going back to the calculus without И, we can recover soundness in nominal sets by substituting rule ($\curlywedge$**var**) with:

$$\frac{\pi^{\rho^{-1}} \in \langle \mathtt{perm}(\Upsilon|_X) \rangle}{\Upsilon \vdash \pi \curlywedge \rho \cdot X} \ (\mathbf{Gvar}_{\curlywedge})$$

However, we lose expressive power because we cannot obtain a fixed-point judgment to mimic $a\#X, b\#X \vdash (a\ b) \cdot [a]X \approx [a]X$.

$$\dfrac{\dfrac{\dfrac{(c_3\ c_1) \in \langle (c_1\ c_2), (c_3\ c_4), (a\ e'), (b\ d') \rangle}{\Upsilon, (c_1\ c_2) \curlywedge X, (c_3\ c_4) \curlywedge X \vdash (a\ c_1) \curlywedge (a\ c_3) \cdot X}}{\Upsilon, (c_1\ c_2) \curlywedge X \vdash (a\ c_1) \curlywedge [a]X} \quad \dfrac{\dfrac{(b\ d_1) \in \langle (d_1\ d_2), (c_3'\ c_4'), (a\ e'), (b\ d') \rangle}{\Upsilon, (d_1\ d_2) \curlywedge X, (c_3'\ c_4') \curlywedge X \vdash (b\ d_1) \curlywedge (a\ c_3') \cdot X}}{\Upsilon, (d_1\ d_2) \curlywedge X \vdash (b\ d_1) \curlywedge [a]X}}{\{(a\ e') \curlywedge X, (b\ d') \curlywedge X\} \vdash (a\ b) \cdot [a]X = [a]X} \ (\mathbf{perm})$$

Conclusion and Future Work

Conclusion and Future Work

Summary:

- Novel semantic interpretation of general fixed-point constraints
- Unlike freshness constraints, fixed-point constraints are not denotationally sound for nominal sets, only for strong nominal sets.
- To recover soundness, we changed our fixed-point context to contain primitive constraints of the form $\text{И}c.(a\ c) \curlywedge X$.
- As a step towards completeness, we proved for strong theory T, $\mathbb{F}(\mathsf{T}, \Sigma)$ is strong nominal.

Future Work:

- Investigate the completeness of fixed-point constraints.
- Extend the calculus with И modulo C.
- Apply the results to unification problems modulo C.

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